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On monotone rational approximation

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Abstract

Let f be an absolutely continuous function on [0, 1] satisfying $f' \in L_p[0,1]$, $p > 1$, Q_n-be the set of all rational functions $r = s/q$, where s and q are polynomials of degree $\leqslant n$. We prove: if f is a *monotone* function on [0, 1], then there is a *monotone* rational function $r \in Q_n$, such that

$$\|f - r\|_{C[0,1]} \leqslant \frac{c(p)}{n} \|f'\|_{L_p[0,1]}, \quad n = 1, 2, \ldots.$$

Keywords: Monotone approximation; Rational approximation; Shape-preserving approximation

1. Introduction

Let P_n be the space of all algebraic polynomials of degree at most n, and Q_n be the set of all rational functions $r = \frac{s}{q}$, where $s, q \in P_n$. The error of the best uniform rational approximation of a continuous function f on [0, 1] is defined by

$$\rho_n(f) = \inf_{r \in Q_n} \|f - r\|_{C[0,1]}.$$

Let Δ^1 be the set of all monotone continuous functions on [0, 1]. For $f \in \Delta^1$ we set

$$\rho_n^{(1)}(f) = \inf_{r \in Q_n \cap \Delta^1} \|f - r\|_{C[0,1]},$$

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the error of the best monotone rational approximation. The estimates of ρ_n for Sobolev classes W_p^r were obtained by R.A. DeVore, A.A. Pekarskii, P.P. Petrushev, V.A. Popov and others (see e.g. [5]). Some analogs of these estimates in the shape-preserving approximation are obtained in [1,6], however it seems that no exact results are known until the present time. In this paper we solve a problem raised by R.A. DeVore in several lectures during the last 15 years. Namely, we find the exact order of monotone rational approximation for the Sobolev classes W_p^1. Recall that a function $g \in L_p[0.1]$, $1 \leqslant p < \infty$, if

$$\|g\|_p := \left(\int_0^1 |g(x)|^p \, dx \right)^{1/p} < +\infty.$$

Our main result is

Theorem 1. *Let* $1 < p < \infty$, *and f be an absolutely continuous function on* $[0, 1]$, *satisfying* $f' \in L_p[0, 1]$. *If f is a monotone function on* $[0, 1]$, *then*

$$\rho_n^{(1)}(f) < \frac{c(p)}{n} \|f'\|_p, \quad n = 1, 2, \ldots, \tag{1}$$

where $c(p)$ *is a constant depending only on p.*

If $p = 1$, then for any sequence $\varepsilon_0 \geqslant \varepsilon_1, \ldots, \lim \varepsilon_n = 0$ there is a monotone absolutely continuous function f such that $\|f'\|_1 \leqslant 1$ and $\rho_n^{(1)}(f) > \varepsilon_n$, see [4, pp. 241–242]. For $p = \infty$ already the approximation by monotone polynomials of degree $\leqslant n$ provides (1) [3]. Finally, note that for each $n \in \mathbb{N}$ there is a function f such that $\|f'\|_p = 1$, and satisfying the conditions of Theorem 1, and for which the opposite to inequality (1) holds. To construct a corresponding example one can easily modify the arguments of Theorem 7.5 in [4]. Namely, the function $f_{n+1}(x) + x$ provides the required estimate for each $n \in \mathbb{N}$, where f_n is defined in [4, p. 240].

In Section 2 we prove some auxiliary results, in Section 3 we prove Main Lemma and in Section 4 we prove Theorem 1.

2. Auxiliary lemmas

We will prove Lemmas 1–6 for each fixed pair $m, n \in \mathbb{N}$ such that m is even and $N_0 \leqslant m < n$, where N_0 is an absolute constant, large enough. Namely, N_0 is a number such that the last inequalities in (9), (10), (16), (17), (19), (20), (36), (52), (57) and (58) hold.

Lemma 1. *The polynomial*

$$T_n(x) := \sum_{k=0}^{3n} (-1)^k n^{2k+1} \frac{x^{2k}}{(2k+1)!} \tag{2}$$

of degree $6n$ *and the number*

$$A := A_{m,n} := \int_{-1}^{1} T_n^m(x)\,dx \tag{3}$$

satisfy

$$\frac{1}{2}\left(\frac{\sin nx}{x}\right)^m \leqslant T_n^m(x) \leqslant 2\left(\frac{\sin nx}{x}\right)^m, \quad x \in \left(0, \frac{3}{n}\right], \tag{4}$$

$$\int_x^1 T_n^m(t)\,dt \leqslant \frac{1}{(m-2)x^{m-1}}, \quad x \in \left[\frac{2}{n}, 1\right], \tag{5}$$

$$A \geqslant \frac{1}{2m}\,n^{m-1}, \tag{6}$$

$$A \leqslant 6n^{m-1}. \tag{7}$$

Proof. First we expand the function

$$g(x) := \frac{\sin nx}{x}$$

$(g(0) := n)$ in the Taylor series and get

$$\begin{aligned} |g(x) - T_n(x)| &= \left|\sum_{k=3n+1}^{\infty} (-1)^k n^{2k+1} \frac{x^{2k}}{(2k+1)!}\right| \leqslant \sum_{k=3n+1}^{\infty} \frac{n^{2k+1}}{(2k+1)!} \\ &\leqslant \sum_{k=3n+1}^{\infty} \frac{(3n)^{2k+1}}{(2k+1)^{2k+1}} \leqslant \sum_{k=3n+1}^{\infty} 2^{-(2k+1)} \\ &< 2^{-6n}, \quad x \in [-1, 1], \end{aligned} \tag{8}$$

where we used the inequality $n! > (n/3)^n$. Since g is decreasing on $(0, \pi/n]$,

$$g(x) \geqslant g(3/n) = n\,\frac{\sin 3}{3} > 1, \quad x \in (0, 3/n]. \tag{9}$$

Hence, for $x \in (0, 3/n]$,

$$1 - 2^{-6n} \leqslant 1 - \frac{|T_n(x) - g(x)|}{g(x)} \leqslant \frac{T_n(x)}{g(x)} \leqslant 1 + \frac{|T_n(x) - g(x)|}{g(x)} \leqslant 1 + 2^{-6n},$$

that implies (4). Then, (8) yields

$$|T_n(x)| \leqslant \frac{1}{x} + 2^{-6n}, \quad x \in (0, 1].$$

Therefore, for $x \in [2/n, 1]$,

$$\int_x^1 T_n^m(t)\,dt \leqslant \int_x^1 \left(\frac{1}{t} + 2^{-6n}\right)^m dt \leqslant \int_x^1 \left(\frac{1}{t^m} + 2^m \frac{2^{-6n}}{t^{m-1}}\right) dt$$
$$\leqslant \frac{1}{(m-1)x^{m-1}} + \frac{2^{m-6n}}{(m-2)x^{m-2}} \leqslant \frac{1}{(m-2)x^{m-1}}, \tag{10}$$

which is (5). Now, to prove (7) we represent A in the form

$$A = 2\int_0^1 T_n^m(x)\,dx = 2\int_0^{2/n} T_n^m(x)\,dx + 2\int_{2/n}^1 T_n^m(x)\,dx.$$

Hence (8) and (5) imply

$$A \leqslant 2 \cdot \frac{2}{n}\left(n + 2^{-6n}\right)^m + 2 \cdot \frac{n^{m-1}}{(m-2)2^{m-1}} \leqslant 6n^{m-1}.$$

Finally, we again apply (8) and get

$$A \geqslant \int_0^{\frac{1}{mn}} \left(g(t) - 2^{-6n}\right)^m dt \geqslant \int_0^{\frac{1}{mn}} \left(mn \sin\frac{1}{m} - 2^{-6n}\right)^m dt$$
$$= \frac{1}{mn}\left(mn \sin\frac{1}{m} - 2^{-6n}\right)^m > \frac{1}{mn} \cdot n^m \left(1 - \frac{1}{3m^2}\right)^m > \frac{1}{2m} n^{m-1},$$

which is (6). Lemma 1 is proved. □

Lemma 2. *Let T_n and A be defined by Lemma 1, and denote*

$$P_m(x) := \sum_{k=0}^{m} (-1)^k n^{2k+1} \frac{x^{2k}}{(2k+1)!} - n^{2m+1} x^{2m} \left(\frac{2}{5}\right)^{2m}, \tag{11}$$

a polynomial of degree $2m$. Then the following inequalities hold:

$$0 \leqslant P_m(x) \leqslant T_n(x), \quad x \in \left[0, \frac{2}{n}\right], \tag{12}$$

$$\frac{2}{A}\int_0^{1/(2n)} P_m^m(x)\,dx \geqslant 1 - 16m\left(\frac{29}{30}\right)^m, \tag{13}$$

$$\frac{2}{A}\int_0^x P_m^m(t)\,dt \leqslant m^m (nx)^{2m^2+1}, \quad x \in \left[\frac{2}{n}, 1\right] \tag{14}$$

and

$$\frac{2}{A}\int_0^x P_m^m(t)\,dt \geqslant \frac{1}{3^m}\left(\frac{2}{5}\right)^{2m^2} (nx)^{2m^2+1}, \quad x \in \left[\frac{3}{n}, 1\right]. \tag{15}$$

Proof. We have

$$T_n(x) - P_m(x) = n^{2m+1}x^{2m}\left(\frac{2}{5}\right)^{2m} + \sum_{k=m+1}^{3n}(-1)^k n^{2k+1}\frac{x^{2k}}{(2k+1)!}$$
$$= n^{2m+1}x^{2m}\left(\left(\frac{2}{5}\right)^{2m} + \sum_{k=m+1}^{3n}(-1)^k n^{2k-2m}\frac{x^{2k-2m}}{(2k+1)!}\right).$$

For $x \in [0, 2/n]$ this yields

$$T_n(x) - P_m(x) \geqslant n^{2m+1}x^{2m}\left(\left(\frac{2}{5}\right)^{2m} - \sum_{k=1}^{\infty}\frac{2^{2k}}{(2k+2m+1)!}\right) \geqslant 0 \tag{16}$$

and

$$T_n(x) - P_m(x) \leqslant 2^{2m}n\left(\left(\frac{2}{5}\right)^{2m} + \sum_{k=1}^{\infty}\frac{2^{2k}}{(2k+2m+1)!}\right)$$
$$\leqslant 2n\left(\frac{4}{5}\right)^{2m} \leqslant T_n(x), \tag{17}$$

where in the last inequality we applied (4). Taking into account (16) and (17), we get (12). Then, (4)–(7) and (17) imply

$$1 - \frac{2}{A}\int_0^{1/(2n)} P_m^m(x)\,dx$$
$$= \frac{2}{A}\int_0^{1/(2n)}\left(T_n^m(x) - P_m^m(x)\right)dx + \frac{2}{A}\int_{1/(2n)}^1 T_n^m(x)\,dx$$
$$= \frac{2}{A}\left(\int_0^{1/(2n)}(T_n(x) - P_m(x))\sum_{k=0}^{m-1}T_n^k(x)P_m^{m-1-k}(x)\,dx\right.$$
$$\left.+ \int_{1/(2n)}^{2/n} T_n^m(x)\,dx + \int_{2/n}^1 T_n^m(x)\,dx\right)$$
$$\leqslant \frac{4m}{n^{m-1}}\left(\int_0^{1/(2n)} 2\left(\frac{4}{5}\right)^{2m} n\cdot mT_n^{m-1}(x)\,dx\right.$$
$$\left.+\frac{3}{2n}\cdot 2(2n\sin 1/2)^m + \frac{n^{m-1}}{(m-2)2^{m-1}}\right)$$
$$\leqslant \frac{4m}{n^{m-1}}\left(\left(\frac{4}{5}\right)^{2m}\cdot m\cdot 2n^{m-1} + 3n^{m-1}\left(\frac{29}{30}\right)^m + \frac{n^{m-1}}{2^{m-1}}\right) \leqslant 16m\left(\frac{29}{30}\right)^m,$$

which is (13). The evident inequality

$$|P_m(x)| < mn^{2m+1}x^{2m}, \quad x \in [2/n, 1], \tag{18}$$

(6), (12) and (3) imply, for $x \in [2/n, 1]$,

$$\begin{aligned}\frac{2}{A}\int_0^x P_m^m(t)\,dt &\leqslant 1 + \frac{2}{A}\int_{2/n}^x \left(mn^{2m+1}t^{2m}\right)^m\,dt \\ &\leqslant 1 + \frac{4m}{n^{m-1}}\cdot m^m \cdot n^{2m^2+m}\,\frac{x^{2m^2+1}}{2m^2+1} \\ &< m^m(nx)^{2m^2+1},\end{aligned}$$

which is (14). Finally, for $x \geqslant 11/(4n)$ we have

$$\begin{aligned}|P_m(x)| &\geqslant n^{2m+1}x^{2m}\left(\frac{2}{5}\right)^{2m} - \sum_{k=0}^{m}\frac{n^{2k+1}x^{2k}}{(2k+1)!} \\ &\geqslant n^{2m+1}x^{2m}\left(\left(\frac{2}{5}\right)^{2m} - \sum_{k=0}^{m}\left(\frac{4}{11}\right)^{2m-2k}\frac{1}{(2k+1)!}\right) \\ &\geqslant n^{2m+1}x^{2m}\left(\left(\frac{2}{5}\right)^{2m} - \frac{1}{6}(m+1)\left(\frac{4}{11}\right)^{2m-2}\right) \\ &\geqslant \frac{1}{2}n^{2m+1}x^{2m}\left(\frac{2}{5}\right)^{2m}.\end{aligned} \tag{19}$$

Thus, (7) yield for $x \in [3/n, 1]$,

$$\begin{aligned}\frac{2}{A}\int_0^x P_m^m(t)\,dt &\geqslant \frac{2}{A}\int_{11/(4n)}^x \left(\frac{1}{2}n^{2m+1}t^{2m}\left(\frac{2}{5}\right)^{2m}\right)^m\,dt \\ &\geqslant \frac{2}{6n^{m-1}}\cdot\frac{1}{2^m}\left(\frac{2}{5}\right)^{2m^2}\frac{n^{2m^2+m}}{2m^2+1}\left(x^{2m^2+1} - \left(\frac{11}{4n}\right)^{2m^2+1}\right) \\ &\geqslant \frac{1}{3^m}\left(\frac{2}{5}\right)^{2m^2}(nx)^{2m^2+1},\end{aligned} \tag{20}$$

which is (15). Lemma 2 is proved. □

We denote by

$$R_m(x) := \frac{N(x) - N(-x)}{N(x) + N(-x)}, \tag{21}$$

the Newman rational function, where

$$N(x) := \prod_{i=1}^{m}(x + a^i) \quad \text{and} \quad a := e^{-1/\sqrt{m}}.$$

To prove the following Lemma 3 one need a minor improvement of the (7) in the paper by Iliev and Opitz [2], namely, that

$$\frac{1}{1-\xi} \leqslant 2\sqrt{n}$$

for sufficiently large n.

Lemma 3. *The function R_m satisfies*

$$|1 - R_m(x)| \leqslant 3e^{-\sqrt{m}}, \quad x \in [e^{-\sqrt{m}}, 1], \tag{22}$$

$$R'_m(x) \geqslant 0, \quad x \in [0, e^{-\sqrt{m}}] \tag{23}$$

and

$$|R'_m(x)| \leqslant 16m^{3/2}, \quad x \in [e^{-\sqrt{m}}, \infty). \tag{24}$$

Put

$$b := 32m^{3/2}e^{-\sqrt{m}/2} \tag{25}$$

and

$$\hat{R}_m(x) := \frac{R_m(nxe^{-\sqrt{m}/2}) + bnx}{R_m(e^{-\sqrt{m}/2}) + b}, \tag{26}$$

where R_m defined by (21). Lemma 3 implies the following

Lemma 4. *The rational function $\hat{R_m}$ is odd and*

$$\hat{R}_m(0) = 0, \quad \hat{R_m}\left(\frac{1}{n}\right) = 1, \tag{27}$$

$$\hat{R}'_m(x) \geqslant 0, \quad x \in [0, 1], \tag{28}$$

$$\hat{R}'_m(x) \leqslant \frac{3b}{2}n, \quad x \in \left[\frac{e^{-\sqrt{m}/2}}{n}, 1\right], \tag{29}$$

$$\hat{R}_m(x) \leqslant 1 + \frac{3b}{2}n\left(x - \frac{1}{n}\right), \quad x \in \left[\frac{1}{n}, 1\right], \tag{30}$$

$$\hat{R}_m\left(\frac{e^{-\sqrt{m}/2}}{n}\right) \geqslant 1 - \frac{3b}{2}. \tag{31}$$

Proof. The function $\hat{R}_m$ is odd, since R_m is odd by its definition (21). Equations (27) readily follow from the definition (26). The estimate (22) yields $R_m(e^{-\sqrt{m}/2}) + b \geqslant 1$. This inequality, (24), (25) and the identity

$$\hat{R}'_m(x) = \frac{ne^{-\sqrt{m}/2}\left(R'_m(nxe^{-\sqrt{m}/2}) + 32m^{3/2}\right)}{R_m(e^{-\sqrt{m}/2}) + b}$$

imply (28) and (29). Finally, the estimates (30) and (31) are consequences of (27) and (29). Lemma 4 is proved. □

Lemma 5. *For the odd rational function*

$$H_{n,m}(x) := \frac{\hat{R}_m(x)}{1+4b} - \frac{2}{A}\int_0^x P_m^m(t)\,dt, \tag{32}$$

the following inequalities hold:

$$H_{n,m}(x) < 1, \quad x \in \left[0, \frac{1}{2n}\right], \tag{33}$$

$$H_{n,m}(x) \leqslant 0, \quad x \in \left[\frac{1}{2n}, \frac{3}{n}\right] \tag{34}$$

and

$$H_{n,m}(x) \leqslant -\frac{1}{A}\int_0^x P_m^m(t)\,dt, \quad x \in \left[\frac{3}{n}, 1\right]. \tag{35}$$

Proof. By (32), (28) and (27) we have

$$H_{n,m}(x) = \frac{\hat{R}_m(x)}{1+4b} - \frac{2}{A}\int_0^x P_m^m(t)\,dt < \frac{\hat{R}_m(x)}{1+4b}$$
$$< \frac{\hat{R}_m(1/n)}{1+4b} = \frac{1}{1+4b} < 1, \quad x \in \left[0, \frac{1}{2n}\right],$$

which is (33). Now we prove (34). If $x \in [1/(2n), 3/n]$, then (32), (28), (13) and (30) imply

$$H_{n,m}(x) = \frac{\hat{R}_m(x)}{1+4b} - \frac{2}{A}\int_0^x P_m^m(t)\,dt < \frac{\hat{R}_m(3/n)}{1+4b} - \frac{2}{A}\int_0^{1/(2n)} P_m^m(t)\,dt$$
$$< \frac{1+3b}{1+4b} - 1 + 16m\left(\frac{29}{30}\right)^m = 16m\left(\frac{29}{30}\right)^m - \frac{b}{1+4b} < 0. \tag{36}$$

Finally, (15) and (30), for $x \in [3/n, 1]$, yield

$$\frac{1}{A}\int_0^x P_m^m(t)\,dt \geqslant \frac{1}{2\cdot 3^m}\left(\frac{2}{5}\right)^{2m^2}(nx)^{2m^2+1}$$
$$\geqslant \frac{1}{2\cdot 3^m}\left(\frac{6}{5}\right)^{2m^2}(nx) > 1 + \frac{3b}{2}n\left(x - \frac{1}{n}\right)$$
$$\geqslant \hat{R}_m(x).$$

This inequality and the definition of $H_{n,m}$ provide (35). Lemma 5 is proved. □

Lemma 6. *The rational function*

$$R_m^*(x) := \frac{1}{1+(nx)^{m^5+m}}$$

is even and satisfies

$$R_m^{*\prime}(x) \leqslant 0, \quad x \in [0, 1], \tag{37}$$

$$0 < R_m^*(x) \leqslant 1, \quad x \in [0, 1], \tag{38}$$

$$1 - R_m^*(x) \leqslant (nx)^{m^5}, \quad x \in \left[0, \frac{1}{2n}\right], \tag{39}$$

$$R_m^*(x) \leqslant \frac{1}{(nx)^{m^5}}, \quad x \in \left[\frac{2}{n}, 1\right], \tag{40}$$

$$- R_m^{*\prime}(x) \leqslant \frac{n}{2^{m^5}}, \quad x \in \left[0, \frac{1}{2n}\right] \tag{41}$$

and

$$\frac{-R_m^{*\prime}(x)}{R_m^*(x)} \geqslant \frac{m^5}{x}, \quad x \in \left[\frac{2}{n}, 1\right]. \tag{42}$$

Proof. The inequalities (37)–(40) are evident. Then, the equalities

$$-R_m^{*\prime}(x) = \frac{(m^5 + m)n(nx)^{m^5+m-1}}{\left(1 + (nx)^{m^5+m}\right)^2}$$

and

$$\frac{-R_m^{*\prime}(x)}{R_m^*(x)} = \frac{(m^5 + m)n(nx)^{m^5+m-1}}{1 + (nx)^{m^5+m}},$$

respectively, imply (41) and (42). Lemma 6 is proved. □

3. Main Lemma

Denote by

$$x_+^0 := \begin{cases} 1, & x \geqslant 0, \\ 0, & x < 0. \end{cases}$$

Put

$$R_{n,m}(x) := \frac{2}{A} \int_0^x T_n^m(t)\, dt + H_{n,m}(x) \cdot \frac{R_m^*(x)}{1+b} \tag{43}$$

and

$$Q_{n,m}(x) := \frac{1 + R_{[n/m]+1,m}(x)}{2}, \tag{44}$$

where $[\cdot]$ denotes the entire part. To prove Theorem 1 we need the following

Main Lemma. *For each even* $m > N_0$ *and integer* $n > m^2$ *we have*

$$Q'_{n,m}(x) \geqslant 0, \quad x \in [-1, 1], \tag{45}$$

$$Q_{n,m} = p_{n,m} + q_{n,m},$$

where

$$p_{n,m} \in \mathscr{P}_{7n}, \quad q_{n,m} \in Q_{2m^6}, \tag{46}$$

$$|Q_{n,m}(x) - x_+^0| < e^{-\sqrt{m}/4}, \quad \frac{e^{-\sqrt{m}/4}}{n} \leqslant |x| \leqslant 1, \tag{47}$$

$$|Q_{n,m}(x) - x_+^0| < \frac{e^{-\sqrt{m}/4}}{j^2}, \quad \frac{j-1}{n} \leqslant |x| \leqslant \frac{j}{n}, \quad j = \overline{2, n} \tag{48}$$

and

$$|Q_{n,m}(x) - x_+^0| < 2, \quad |x| \leqslant 1. \tag{49}$$

Proof. By its definition (43), $R_{n,m}$ is an odd function. Therefore to check (45) we have to prove that

$$R'_{n,m}(x) \geqslant 0, \quad x \in [0, 1]. \tag{50}$$

In accordance with (43) and (32),

$$\begin{aligned} R'_{n,m} &= \frac{2}{A} T_n^m + H'_{n,m} \cdot \frac{R_m^*}{1+b} + H_{n,m} \cdot \frac{R_m^{*\prime}}{1+b} \\ &= \frac{2}{A}\Big(T_n^m - P_m^m\Big)\frac{R_m^*}{1+b} + \frac{2}{A} \cdot \frac{bT_n^m}{1+b} + H_{n,m} \cdot \frac{R_m^{*\prime}}{1+b}, \\ &\quad + \frac{2\,T_n^m}{A(1+b)}(1 - R_m^*) + \frac{\hat{R}'_m}{1+4b} \cdot \frac{R_m^*}{1+b}, \end{aligned}$$

where the last line is nonnegative on [0, 1] by (38) and (28). That is,

$$\begin{aligned} R'_{n,m}(x) \geqslant\ & \frac{2}{A}\Big(T_n^m(x) - P_m^m(x)\Big)\frac{R_m^*(x)}{1+b} \\ & + \frac{2}{A} \cdot \frac{b}{1+b}\, T_n^m(x) + H_{n,m}(x) \cdot \frac{R_m^{*\prime}(x)}{1+b}, \quad x \in [0, 1]. \end{aligned} \tag{51}$$

Now we prove (50) separately on the intervals

$$I_1 := \left[0, \frac{1}{2n}\right], \quad I_2 := \left[\frac{1}{2n}, \frac{2}{n}\right], \quad I_3 := \left[\frac{2}{n}, \frac{3}{n}\right] \quad \text{and} \quad I_4 := \left[\frac{3}{n}, 1\right].$$

If $x \in I_1$, then $H_{n,m} < 1$ by (33). Hence (51), (12), (38) and (37) imply

$$R'_{n,m}(x) \geqslant \frac{2}{A} \cdot \frac{b}{1+b} T_n^m(x) + R_m^{*\prime}(x).$$

Therefore, (7), (4), (41) and (25) yield

$$\begin{aligned} R'_{n,m}(x) &\geqslant \frac{2}{6n^{m-1}} \cdot \frac{b}{1+b} \cdot \frac{1}{2} \left(\frac{\sin nx}{x} \right)^m - \frac{n}{2^{m^5}} \\ &> \frac{b}{7n^{m-1}} \left(2n \sin \frac{1}{2} \right)^m - \frac{n}{2^{m^5}} \\ &> n \left(\frac{b}{7} 2^{-m} - 2^{-m^5} \right) > 0, \quad x \in I_1. \end{aligned}$$

If $x \in I_2$ then the inequality $R'_{n,m}(x) \geqslant 0$ readily follows from (51), (12), (38), (34) and (37). If $x \in I_3$, then $H_{n,m}(x) R^{*\prime}_m(x) \geqslant 0$ by (34) and (37). Hence (51) yields

$$\begin{aligned} R'_{n,m}(x) &\geqslant -\frac{2}{A} P^m_m(x) \cdot \frac{R^*_m(x)}{1+b} + \frac{2}{A} \cdot \frac{b}{1+b} T^m_n(x) \\ &= \frac{2}{A(1+b)} \left(b T^m_n(x) - P^m_m(x) R^*_m(x) \right). \end{aligned}$$

Therefore (4), (18) and (40) imply

$$\begin{aligned} R'_{n,m}(x) &\geqslant \frac{2}{A(1+b)} \left(b \cdot \frac{1}{2} \left(\frac{n \sin 3}{3} \right)^m - \frac{m^m n^{2m^2+m} x^{2m^2}}{(nx)^{m^5}} \right) \\ &\geqslant \frac{2n^m}{A(1+b)} \left(\left(\frac{1}{30} \right)^m - \frac{m^m}{2^{m^5-2m^2}} \right) > 0, \quad x \in I_3. \end{aligned} \tag{52}$$

Finally, (51), (35)–(38) imply, for $x \in I_4$,

$$\begin{aligned} R'_{n,m}(x) &\geqslant -\frac{2}{A} P^m_m(x) \cdot \frac{R^*_m(x)}{1+b} + H_{n,m}(x) \cdot \frac{R^{*\prime}_m(x)}{1+b} \\ &\geqslant -\frac{2}{A} P^m_m(x) \cdot \frac{R^*_m(x)}{1+b} - \frac{1}{A} \cdot \int_0^x P^m_m(t)\, dt \cdot \frac{R^{*\prime}_m(x)}{1+b} \\ &= \frac{R^*_m(x)}{A(1+b)} \left(\frac{-R^{*\prime}_m(x)}{R^*_m(x)} \int_0^x P^m_m(t)\, dt - 2 P^m_m(x) \right) \\ &\geqslant \frac{R^*_m(x)}{A(1+b)} \left(\frac{m^5}{x} \int_0^x P^m_m(t)\, dt - 2 P^m_m(x) \right), \end{aligned}$$

where in the last line we used (42). Since $\int_0^x P^m_m(t)\, dt$ is a positive polynomial of degree $2m^2 + 1$, nondecreasing on $[0, 1]$, then applying Markov inequality for the interval $[0, x]$ we get

$$R'_{n,m}(x) \geqslant \frac{R^*_m(x)}{A(1+b)} \left(\frac{m^5}{x} \int_0^x P^m_m(t)\, dt - 2(2m^2+1)^2 \cdot \frac{1}{x} \int_0^x P^m_m(t)\, dt \right) \geqslant 0.$$

So, (50) and hence (45) is proved. Then, (46) readily follows from (43), (44), (2), (32), (26), (11) and the definition of R^*_m in Lemma 6.

Now we prove the estimates

$$0 < 1 - R_{n,m}(x) < 8b, \quad x \in \left[\frac{e^{-\sqrt{m}/2}}{n}, 1 \right] \tag{53}$$

and

$$0 < 1 - R_{n,m}(x) < \frac{5}{(nx)^{m-1}}, \quad x \in \left[\frac{3}{n}, 1\right]. \tag{54}$$

To this end we note that (43), (3), (34) and (38) imply

$$R_{n,m}(1) = 1 + H_{n,m}(1)\,\frac{R_m^*(1)}{1+b} < 1.$$

Therefore (50) yields

$$R_{n,m}(x) < R_{n,m}(1) < 1, \quad x \in [0, 1],$$

that is, the left-hand sides of (53) and (54) are verified. On the other hand, in accordance with (43) and (32),

$$R_{n,m}(x) = \frac{\hat{R}_m(x)}{1+4b} \cdot \frac{R_m^*(x)}{1+b} + \frac{2}{A}\int_0^x \left(T_n^m(t) - P_m^m(t)\right) dt + \left(1 - \frac{R_m^*(x)}{1+b}\right)\frac{2}{A}\int_0^x P_m^m(t)\, dt.$$

Hence (12) and (38) provide

$$R_{n,m}(x) \geqslant \frac{\hat{R}_m(x)}{1+4b} \cdot \frac{R_m^*(x)}{1+b}, \quad x \in \left[0, \frac{2}{n}\right].$$

Thus, taking into account (31), (39) and the monotonicity of $R_{n,m}$, we get

$$\begin{aligned} R_{n,m}(x) &\geqslant \frac{1}{(1+b)(1+4b)}\hat{R}_m\left(\frac{e^{-\sqrt{m}/2}}{n}\right) \cdot R_m^*\left(\frac{e^{-\sqrt{m}/2}}{n}\right) \\ &\geqslant \frac{1}{(1+b)(1+4b)}\left(1 - \frac{3b}{2}\right)\left(1 - 2^{-m^5}\right) > 1 - 8b,\ x \in \left[\frac{e^{-\sqrt{m}/2}}{n}, 1\right]. \end{aligned}$$

This implies (53). Then, for $x \in [3/n, 1]$, using (6), (5), (32), (14) and (40), we obtain

$$\begin{aligned} 1 - R_{n,m}(x) &= \frac{2}{A}\int_x^1 T_n^m(t)\, dt - H_{n,m}(x) \cdot \frac{R_m^*(x)}{1+b} \\ &\leqslant \frac{4m}{n^{m-1}} \cdot \frac{1}{(m-2)x^{m-1}} + \frac{2}{A}\int_0^x P_m^m(t)\, dt \cdot \frac{1}{(nx)^{m^5}} \\ &\leqslant \frac{4m}{m-2} \cdot \frac{1}{(nx)^{m-1}} + \frac{m^m}{(nx)^{m^5-2m^2-1}} \leqslant \frac{5}{(nx)^{m-1}}. \end{aligned}$$

So, (54) is proved as well. Note that (44), (53) and (54) lead to

$$0 < 1 - Q_{n,m}(x) < 4b, \quad x \in \left[\frac{me^{-\sqrt{m}/2}}{n}, 1\right] \tag{55}$$

and

$$0 < 1 - Q_{n,m}(x) < \frac{5m^{m-1}}{(nx)^{m-1}}, \quad x \in \left[\frac{3m}{n}, 1\right]. \tag{56}$$

Since $Q_{n,m}(-x) = 1 - Q_{n,m}(x)$, $x \in [0, 1]$, then (55) readily implies (47). Moreover, (55) and the inequality

$$4b < \frac{e^{-\sqrt{m}/4}}{(3m)^2}, \tag{57}$$

imply (48) for $j = \overline{1, 3m}$. By (56) we get, for $x \in \left[\frac{j-1}{n}, \frac{j}{n}\right]$, $j = \overline{3m+1, n}$,

$$0 < 1 - Q_{n,m}(x) < \frac{5m^{m-1}}{(j-1)^{m-1}} < \frac{e^{-\sqrt{m}/4}}{j^2}. \tag{58}$$

This provides (48). Finally, the inequality (49) readily follows from (47), (48) and monotonicity of $Q_{n,m}$. The Main Lemma is proved. □

4. Proof of Theorem 1

Without any loss of generality we assume that f is a nondecreasing function on [0, 1], and $\|f'\|_p = 1$. Given $n \in \mathbb{N}$, let us consider the partition $0 = x_0 < x_1 < \cdots < x_k \leqslant 1$, satisfying

$$f(x_{i+1}) - f(x_i) = \frac{1}{n}, \quad 1 = \overline{0, k-1},$$

$$f(1) - f(x_k) < \frac{1}{n}.$$

By Hölder inequality we have

$$\left(\int_{x_i}^{x_{i+1}} (f'(x))^p \, dx\right)^{1/p} (x_{i+1} - x_i)^{\frac{p-1}{p}} \geqslant \int_{x_i}^{x_{i+1}} f'(x)\, dx = \frac{1}{n}, \quad i = \overline{0, k-1},$$

hence

$$\int_{x_i}^{x_{i+1}} (f'(x))^p \, dx \geqslant \frac{1}{n^p (x_{i+1} - x_i)^{p-1}} \quad \text{for } i = \overline{0, k-1}.$$

This inequality and the assumption $\|f'\|_{L_p} = 1$ imply

$$\sum_{i=0}^{k-1} \frac{1}{(n(x_{i+1} - x_i))^{p-1}} \leqslant n, \tag{59}$$

whence

$$\frac{1}{n(x_{i+1} - x_i)} \leqslant n^{\frac{1}{p-1}}, \quad i = \overline{0, k-1}. \tag{60}$$

Note that Hölder inequality also implies

$$\frac{k}{n} \leqslant f(1) - f(0) = \|f'\|_{L_1} \leqslant \|f'\|_{L_p} = 1,$$

whence $k \leqslant n$. Put

$$S(x) = f(x_0) + \frac{1}{n}\sum_{i=1}^{k-1}(x - x_i)_+^0.$$

Evidently $S(x_i) = f(x_i)$, for all $i = \overline{0, k-1}$, hence

$$\|S - f\|_{C[0,1]} \leqslant \frac{2}{n}.$$

Now, for each $i = \overline{1, k-1}$ set

$$m_i := 2N_0 + 16[\ln_+^2(n^{-1}\max\{(x_{i+1} - x_i)^{-1}, (x_i - x_{i-1})^{-1}\})] \tag{61}$$

and note that m_i are even, $m_i > N_0$,

$$e^{-\sqrt{m_i}/4} < n \min\left\{x_{i+1} - x_i, x_i - x_{i-1}, \frac{1}{n}\right\}, \tag{62}$$

thus, (60) yields

$$m_i \leqslant 2N_0 + 16 \ln^2 n^{\frac{1}{p-1}}.$$

Given $p > 1$ let $N(p)$ be so that if $n > N(p)$, then

$$\left(2N_0 + 16 \ln^2 n^{\frac{1}{p-1}}\right)^2 < n.$$

Take $n > N(p)$ (if $n \leqslant N(p)$, Theorem 1 is evident). Then $n > m_i^2$ for all $i = \overline{1, k-1}$, so we may use the Main Lemma for the rational functions Q_{n,m_i} defined by (44). Put

$$R(x) := f(x_0) + \frac{1}{n}\sum_{i=1}^{k-1} Q_{n,m_i}(x - x_i).$$

Since each Q_{n,m_i} is a nondecreasing function, so is R. For each fixed $x \in [0, 1]$ we have

$$\begin{aligned} n|f(x) - R(x)| &\leqslant n|f(x) - S(x)| + n|S(x) - R(x)| \\ &\leqslant 2 + \sum_{i=1}^{k-1} |(x - x_i)_+^0 - Q_{n,m_i}(x - x_i)| \\ &= 2 + \sum_{i:|x-x_i| \leqslant \frac{1}{n} e^{-\sqrt{m_i}/4}} |(x - x_i)_+^0 - Q_{n,m_i}(x - x_i)| \\ &\quad + \sum_{i:\frac{1}{n} e^{-\sqrt{m_i}/4} < |x-x_i| \leqslant \frac{1}{n}} |(x - x_i)_+^0 - Q_{n,m_i}(x - x_i)| \\ &\quad + \sum_{j=2}^{n} \sum_{i:\frac{j-1}{n} < |x-x_i| \leqslant \frac{j}{n}} |(x - x_i)_+^0 - Q_{n,m_i}(x - x_i)| \end{aligned}$$

$$\leqslant 2+\sum_{i:|x-x_i|\leqslant \frac{1}{n}e^{-\sqrt{m_i}/4}} |(x-x_i)_+^0 - Q_{n,m_i}(x-x_i)| + \sum_{j=1}^{n} \sum_{i:\frac{j-1}{n}<|x-x_i|\leqslant \frac{j}{n}} \frac{1}{j^2} e^{-\sqrt{m_i}/4}, \tag{63}$$

where we used (47) and (48) in the last inequality. By (62), we get $|x_i - x_{i\pm 1}| > n^{-1}e^{-\sqrt{m_i}/4}$, whence there are at most two indices i, satisfying $|x-x_i| < n^{-1}e^{-\sqrt{m_i}/4}$. Therefore (49) yields

$$\sum_{i:|x-x_i|\leqslant \frac{1}{n}e^{-\sqrt{m_i}/4}} |(x-x_i)_+^0 - Q_{n,m_i}(x-x_i)| < 4. \tag{64}$$

Thus, (62)–(64) provide

$$|f(x)-R(x)|\leqslant \frac{6}{n} + \sum_{j=1}^{n} \frac{1}{j^2} \sum_{i:\frac{j-1}{n}<|x-x_i|\leqslant \frac{j}{n}} \min\left\{(x_{i+1}-x_i), \frac{1}{n}\right\}. \tag{65}$$

Since, $x_i < x_{i+1}$, $i=\overline{0, k-1}$, then

$$\sum_{i:\frac{j-1}{n}<|x-x_i|\leqslant \frac{j}{n}} \min\left\{x_{i+1}-x_i, \frac{1}{n}\right\} = \sum_{i:x+\frac{j-1}{n}<x_i\leqslant x+\frac{j}{n}} \min\left\{x_{i+1}-x_i, \frac{1}{n}\right\} + \sum_{i:x-\frac{j}{n}\leqslant x_i<x-\frac{j-1}{n}} \min\left\{x_{i+1}-x_i, \frac{1}{n}\right\} \leqslant \frac{2}{n}+\frac{2}{n}=\frac{4}{n}.$$

By (65) we get

$$|f(x)-R(x)|\leqslant \frac{6}{n}+\frac{1}{n}\sum_{j=1}^{n} 4\cdot\frac{1}{j^2} < \frac{13}{n}. \tag{66}$$

Finally, (46) and (59) imply

$$\deg(R) \leqslant 7n + 2\sum_{i=1}^{k-1} m_i^6 \leqslant 7n + 4\sum_{i=0}^{k-1}\left(2N_0 + 16\ln_+^2 \frac{1}{n(x_{i+1}-x_i)}\right)^6 \leqslant 7n + 4\sum_{i=0}^{k-1} c(p)\left(1+\frac{1}{(n(x_{i+1}-x_i))^{p-1}}\right) \leqslant n(7+8c(p)),$$

where $c(p)$ is a constant such that

$$(2N_0 + 16\ln_+^2 x)^6 \leqslant c(p)(1+x^{p-1}), \quad x>0.$$

Combining (66) and the last inequality we obtain (1), which completes the proof of the Theorem 1. □

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